Practice Midterm Question 4b: When I first saw this question I thought there was a typo and we would have needed to find a **not empty** subspace W. Here is how I answered this question finding a not empty subset.

Recall the problem:

Question 4b: Let $V = \{p \in \mathcal{P}_3(\mathbb{R}) | p'(5) = 0\}$, find a subspace W such that $V + W$ is a direct sum, in other words we need to find a subspace such that $V \bigcap W = \{\mathbf{0}\}$.

I used an idea from 6A together with the isomorphism between $\mathbb{R}^4$ and $\mathcal{P}_3(\mathbb{R})$.
I will proceed as follow:

Step 1: I will first explain how there is an Isomorphism between $\mathbb{R}^4$ and $\mathcal{P}_3(\mathbb{R})$.

Step 2: I will then convert V in $\mathbb{R}^4$ using this identification and I notice that it is a three dimensional subspace in $\mathbb{R}^4$.

Step 3: Three dimensional subspaces of $\mathbb{R}^4$ are hyperplane, I can write an hyperplane as $\mathbf{n} \cdot \mathbf{x} = 0$ (as we saw in 6A), where $\mathbf{n}$ is the normal vector, then I can find the orthogonal line (parallal to the normal) and that would be a 1 dimensional subspace orthogonal to the hyperplane, therefor their intersection is just the zero vector.

Step 4: I can check that the intersection in $\mathbb{R}^4$ is only the zero vector.

Step 5: I will now apply the inverse of the isomorphism found in step 2 to find my W

Step 1 Define f as follows: $f : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^4$ where

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mapsto (a_0, a_1, a_2, a_3)$$

Its very easy to see that this is a isomorphism since it sends the basis $\{1, x, x^2, x^3\}$ to the standard basis of $\mathbb{R}^4$.

Step 2 The condition given in V is $p'(5) = 0$, so I compute: let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, therefor $p'(x) = a_1 + 2a_2 x + 3a_3 x^2$, I now plug five and obtain:

$$a_1 + 10a_2 + 75a_3 = 0$$

Step 3 I notice that now under the isomorphism f I have an hyperplan $H = f(V)$, i.e.:

$$H = \{(a_0, a_1, a_2, a_3) \in \mathbb{R}^4 | a_1 + 10a_2 + 75a_3 = 0\}$$

We know from 6A that hyperplane in euclidean space can be written using the dot product, i.e. let $\mathbf{a} = (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$ and let $n \in \mathbb{R}^4$ be another vector who we called the normal vector. then any hyperplane can be written as $\mathbf{n} \cdot \mathbf{a} = 0$ for some $\mathbf{n}$, Clearly for our H the correct choice for n (recall that $\mathbf{n} \cdot \mathbf{a} = n_0 a_0 + n_1 a_1 + n_2 a_2 + n_3 a_3$) is:

$$H = \mathbf{n} \cdot \mathbf{a} \text{ with } \mathbf{n} = (0, 1, 10, 75)$$

Step4 Now it is geometrically obvious that the normal to the hyperplane and the hyperplan only intersect at origin. So I can now define the one dimensional subspace, i.e. a line that goes in the direction of **n**:

$$L = \{(0, t, 10t, 75t) \in \mathbb{R}^4 | t \in \mathbb{R}\}$$

Step 5 I can now pull back L with the inverse of the isomorphism f and obtain my non zero W:

$$W = f^{-1}(L) = \{tp(x) \in \mathcal{P}_3(\mathbb{R}) | p(x) = x + 10x^2 + 75x^3 \text{ and } t \in \mathbb{R}\}$$

Notice that the isomorphism preseverses dimensions and intersection of subspaces so we have found a non zero subspace W for which $\mathcal{P}_3(\mathbb{R})$ can be written as direct sum of W and V. W is one dimensional and V is three dimensional, the only subspace of $\mathcal{P}_3(\mathbb{R})$ that is 4 dimensional is itself, we then conclude:

$$\mathcal{P}_3(\mathbb{R}) = V \bigoplus W.$$